*Proc. Nat. Acad. Sci. USA*
Vol. 73, No. 2, pp. 283–285, February 1976
Physics

# On the lifetime-width relation for a decaying state and the uncertainty principle

**(decay law/line shape/characteristic time/equivalent width/time coherence)**

M. BAUER AND P. A. MELLO

Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20–364, México 20, D.F.

*Communicated by Eugene P. Wigner, October 9, 1975*

**ABSTRACT** **A new formulation of the uncertainty relation of position and momentum, and of energy and time, is presented. The connection between the lifetime of excited states and the energy width of these states, which does not follow from the usual uncertainty relation, is shown to be a consequence of the expression here derived.**

The relation $\Gamma\tau = 1$ ($\hbar = 1$) between the width $\Gamma$ and the lifetime $\tau$ of a decaying state was derived long ago by Wigner and Weisskopf (ref. 1). The main features of the derivation can be obtained in the following schematic model. The state $|i\rangle$ is coupled at time $t = 0$ to an infinite discrete spectrum of equally spaced levels $|n\rangle$. We assume for simplicity $E_i = 0$ and consider the coupling matrix elements $V_n = \langle i|V|n\rangle$ to be independent of $n$: $V_n = V$. Then, taking the limit of the level spacing $\delta$ going to zero ($|n\rangle \rightarrow |E\rangle$) subject to the condition $2\pi V^2/\delta = \gamma$, where $\gamma$ is a constant, it is easily seen that

$$|\psi(t)\rangle = \mathcal{C}_i(t)|i\rangle + \int dE\, \mathcal{C}_E(t)|E\rangle\ , t \geq 0, \qquad [1]$$

with†

$$\mathcal{C}_i(t) = e^{-(\gamma/2)t}\theta(t), \qquad [2]$$

where $\theta(t)$ is the step function and

$$\mathcal{C}_E(t) = -i\sqrt{\frac{\gamma}{2\pi}}\, e^{-iEt}\int_0^t \mathcal{C}_i(t')e^{iEt'}dt'. \qquad [3]$$

The line shape is given by

$$\left|\mathcal{C}_E(t=\infty)\right|^2 = \gamma\left|\frac{1}{\sqrt{2\pi}}\int_0^\infty \mathcal{C}_i(t')\,e^{iEt'}dt'\right|^2 = \frac{\gamma/(2\pi)}{E^2 + (\gamma/2)^2} \qquad [4]$$

which is a Lorentzian of width at half-maximum $\Gamma$ given by the constant $\gamma$. On the other hand, Eq. 2 gives an exponential decay law for the initial state, with a lifetime $\tau$ equal to $1/\gamma$, so that $\Gamma\tau = 1$.

The usual claim is that this result is just the manifestation of the underlying time-energy uncertainty relation. We proceed to test this claim by applying the two existing formulations of the time-energy uncertainty principle to the present case.

† This is the mathematical limit of the situation in which the state $|i\rangle$ is populated much more rapidly than it decays.

## (*1*) The Mandelštam–Tamm formulation (refs. 2 and 3)

For any hermitian operator $\hat{A}$ and any state vector $|\psi(t)\rangle$ one can define a "characteristic time"

$$\tau_A \equiv (\Delta\hat{A})\left|\frac{d\langle\hat{A}\rangle}{dt}\right|^{-1}, \qquad [5]$$

where $\langle\hat{A}\rangle$ is the expectation value of $\hat{A}$ in the state $|\psi(t)\rangle$ and $\Delta\hat{A} = [\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2]^{1/2}$ is the corresponding uncertainty. It can then be shown from the uncertainty relation between any two noncommuting operators and from the dynamical equation, that the following inequality holds:

$$\tau_A\cdot\Delta\hat{H} \geq \frac{1}{2}, \qquad [6]$$

where $\hat{H}$ is the Hamiltonian of the system. Note that $\tau_A$ is a quantity with dimensions of time related to the *instantaneous* expectation value of the operator $\hat{A}$.

In our problem we choose $\hat{A} = |i\rangle\langle i|$, so that $\langle\hat{A}\rangle = |\mathcal{C}_i(t)|^2$ and

$$\tau_A(t) = \frac{1}{\gamma}(e^{\gamma t}-1)^{1/2}, \qquad [7]$$

which is seen to go from 0 to $\infty$, thus bearing no relation to $\tau$. The inequality [6] is satisfied at all times because $\Delta\hat{H}$ is infinite, as can be seen from the fact that $(\Delta\hat{H})^2 = \int|V(E)|^2\rho(E)dE = \gamma/(2\pi)\int dE = \infty$, being therefore unrelated to $\Gamma$.

It could be argued that a realistic $V(E)$ would in general decrease for distant states and $\Delta\hat{H}$ could be finite. Nevertheless, $(\Delta\hat{H})^2$ is by definition the area under the curve $|V(E)|^2\rho(E)$, of which $\gamma$ is the central ordinate, so that $\Delta H$ is not a measure of $\Gamma$.

## (*2*) The Wigner formulation (ref. 4)

Starting from the projection

$$\mathcal{X}_u(t) = \langle u|\psi(t)\rangle \qquad [8a]$$

of the state vector $|\psi(t)\rangle$ onto an arbitrary vector $|u\rangle$, and its Fourier transform

$$\tilde{\mathcal{X}}_u(E) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{X}_u(t)\,e^{iEt}dt \qquad [8b]$$

and in complete analogy with the formulation of the position-momentum uncertainty relation, Wigner introduces the time and energy spreads

$$(\Delta t)^2 = \frac{\int (t-\bar{t})^2 |\mathfrak{X}_u(t)|^2 dt}{\int |\mathfrak{X}_u(t)|^2 dt} \quad [9a]$$

and

$$(\Delta E)^2 = \frac{\int (E-\bar{E})^2 |\tilde{\mathfrak{X}}_u(E)|^2 dE}{\int |\tilde{\mathfrak{X}}_u(E)|^2 dE}, \quad [9b]$$

which satisfy the relation $\Delta t \cdot \Delta E > \frac{1}{2}$, as he shows that the lower bound on the energy spectrum precludes the equality. Note that one is now integrating over time in contrast with the usual expectation values in which the time is kept fixed.

Taking $|u\rangle = |i\rangle$, we have $\mathfrak{X}_u(t) = \mathcal{C}_i(t)$. Then Eqs. **2** and **9** yield $\Delta t = 1/\gamma$, which is precisely the lifetime. On the other hand, the Fourier transform $\tilde{\mathfrak{X}}_u(E)$ of $\mathfrak{X}_u(t)$ is

$$\tilde{\mathfrak{X}}_u(E) = \tilde{\mathcal{C}}_i(E) = \frac{i/\sqrt{2\pi}}{E + i\gamma/2}. \quad [10]$$

It is important to note (see Eq. 3) that $\tilde{\mathcal{C}}_i(E)$ is proportional to $\lim_{t\to\infty} [\exp(iEt)\mathcal{C}_E(t)]$, whose square coincides precisely with the line shape. This is a general result, not restricted to the present model.

Then from Eq. **10** we see that $(\Delta E)^2$ is the second moment of a Lorentzian and is therefore infinite. This was pointed out long ago by Fock and Krilov (refs. 5 and 6)[‡].

We have thus seen that neither of the existing precise formulations of the time-energy uncertainty relation yields the lifetime-width relation for a decaying state.

**The Equivalent Width.** The variance is not the only way to measure the width of a distribution, and sometimes, as we have seen in the case of a Lorentzian curve, it is totally unrelated. We introduce now the concept of "equivalent width" (ref. 7), which is very familiar in electrical engineering, and show that it can be used as the basis for the relationship between lifetime and width. The equivalent width $W(f)$ of a function $f(u)$ is defined as

$$W(f) = \frac{1}{f(0)} \int_{-\infty}^{\infty} f(u)\,du, \quad [11]$$

provided the integral exists and $f(0)$ is different from zero. Then if $\tilde{f}(v)$ is the Fourier transform of $f(u)$, it can be easily proved that

$$W(f)\cdot W(\tilde{f}) = 2\pi. \quad [12]$$

This is the fundamental relation that the concept of equivalent width provides. We note it is an *equality* as opposed to the usual uncertainty relation.

We now apply the concept of equivalent width to Wigner's energy probability distribution $|\tilde{\mathfrak{X}}_u(E)|^2$, Eq. **8b**. The Fourier transform of this function is given by the autocorrelation function[§]

$$[\mathfrak{X}_u \times \mathfrak{X}_u](t) \equiv \int \mathfrak{X}_u^*(t')\mathfrak{X}_u(t'+t)\,dt', \quad [13]$$

where $\mathfrak{X}_u(t)$ is the Fourier transform of $\tilde{\mathfrak{X}}_u(E)$. Taking $|u\rangle = |i\rangle$ as before, $\mathfrak{X}_u(t) = \mathcal{C}_i(t)$, Eq. **2**, so that

$$[\mathfrak{X}_u \times \mathfrak{X}_u](t) = \int \mathcal{C}_i^*(t')\mathcal{C}_i(t'+t)\,dt' = \frac{1}{\gamma} e^{-(\gamma/2)|t|}, \quad [14]$$

which is a measure of the coherence in time of the wave function of the initial state. Its equivalent width is given by

$$W(\mathfrak{X} \times \mathfrak{X}) = \frac{4}{\gamma} = 4\tau, \quad [15]$$

a quantity of the order of the lifetime $\tau$. On the other hand, $|\tilde{\mathfrak{X}}_u(E)|^2$ is now $|\tilde{\mathcal{C}}_i(E)|^2$, which in turn is proportional to the Lorentzian line shape $|\mathcal{C}_E(t=\infty)|^2$, as seen from Eq. **4**. Its equivalent width is given by

$$W(|\tilde{\mathfrak{X}}|^2) = W(|\mathcal{C}_E(t=\infty)|^2) = \frac{\pi}{2}\gamma = \frac{\pi}{2}\Gamma, \quad [16]$$

a *finite* quantity of the order of the width at half-maximum $\Gamma$. Substituting [**15**] and [**16**] in Theorem **12**, one then obtains $\Gamma\tau = 1$.

Notice that in the definition of the equivalent width [**11**], the value of the function at the origin occurs in the denominator. The autocorrelation function [**13**] has always a positive definite maximum at $t = 0$ for any $\mathfrak{X}_u(t)$. The function $|\tilde{\mathfrak{X}}_u(E)|^2$, which in the previous example is the Lorentzian [**4**], has its maximum precisely at the origin; however, if the energy $E_i$ of the initial state is shifted to some nonzero value, this is no longer true and we lose the simple relation [**16**] between the equivalent width and the width at half-maximum. However, we can modify definition [**11**] of the equivalent width, to express it in terms of an arbitrary value $u_0$ of the variable

$$W_{u_0}(f) \equiv \frac{1}{f(u_0)} \int_{-\infty}^{\infty} f(u)\,du. \quad [17]$$

In terms of this quantity we can now write the fundamental theorem as

$$W_{E_0}(|\tilde{\mathfrak{X}}_u|^2)\cdot W_0(\mathfrak{X}_u e^{iE_0t} \times \mathfrak{X}_u e^{iE_0t}) = 2\pi, \quad [18]$$

where $E_0$ is arbitrary and can be taken as the resonance energy.

Finally, one might still have the reservation that the second W in Eq. **18** applies to a function that is not necessarily real and positive definite. Therefore its equivalent width might not represent an appropriate measure of the physical spread associated with the absolute value square of the function. However, one can easily show that

$$W_0(\mathfrak{X} \times \mathfrak{X}) \le W_0(|\mathfrak{X}| \times |\mathfrak{X}|), \quad [19]$$

so that one can write the inequality

$$W_{E_0}(|\tilde{\mathfrak{X}}_u(E)|^2)\cdot W_0(|\mathfrak{X}_u| \times |\mathfrak{X}_u|) \geqslant 2\pi, \quad [20]$$

containing only absolute values, which are the measurable quantities. The equality holds whenever $\mathfrak{X}_u(t)l^{iE_0t}$ is real and positive, as in the case we treated earlier.

Summarizing, we have shown that the lifetime-width relation is not a consequence of the two known presentations of the time-energy uncertainty principle. On the other hand,

‡ The divergence arises from the Lorentzian tail which is produced by the step function in Eq. 2, i.e., the sudden population of the state $|i\rangle$. A fast but finite rate of population would modify the $E$-dependence of the tails and yield a finite second moment. This one, however, will not be in general a good measure of the width at half-maximum.

§ This is known as the theorem of Wiener–Khinchin.

we have seen that the quantum dynamics $\hat{H} = i\partial/\partial t$, which motivates the Wigner formulation, yields a Fourier type relation between the initial-state amplitude and the probability amplitude of the final-state distribution. This enables us to use the general theorem, Eq. **12**, relating the equivalent widths of a function and its Fourier transform. This relation can be taken as an alternative formulation of the complementary character of time and energy, with the additional feature of being an equality, in contrast with the usual presentation of the uncertainty principle.

This work was supported by the Instituto Nacional de Energía Nuclear, México.

1. Weisskopf, V. & Wigner, E. P. (1930) *Z. Physi* **63**, 54.
2. Mandelstam, L. & Tamm, I. G. (1945) *J. Phys. U.,S.S.R.* **9**, 249.
3. Messiah, A. (1972) in *Quantum Mechanics*, tr. Temmer, G. M. (North-Holland Publishing Co., Amsterdam), pp. 320.
4. Wigner, E. P. (1972) in *Aspects of Quantum Theory*, ed. Salam, A. & Wigner, E. P. (Cambridge University Press), pp. 237.
5. Fock, V. & Krylov, N. (1947) *J. Phys. U.S.S.R.* **11**, 112.
6. Allcock, G. R. (1969) *Ann. Phys.* (*N.Y.*) **53**, 253.
7. Bracewell, R. M. (1965) in *The Fourier Transform and its Applications* (McGraw-Hill Book Co.), pp. 148.